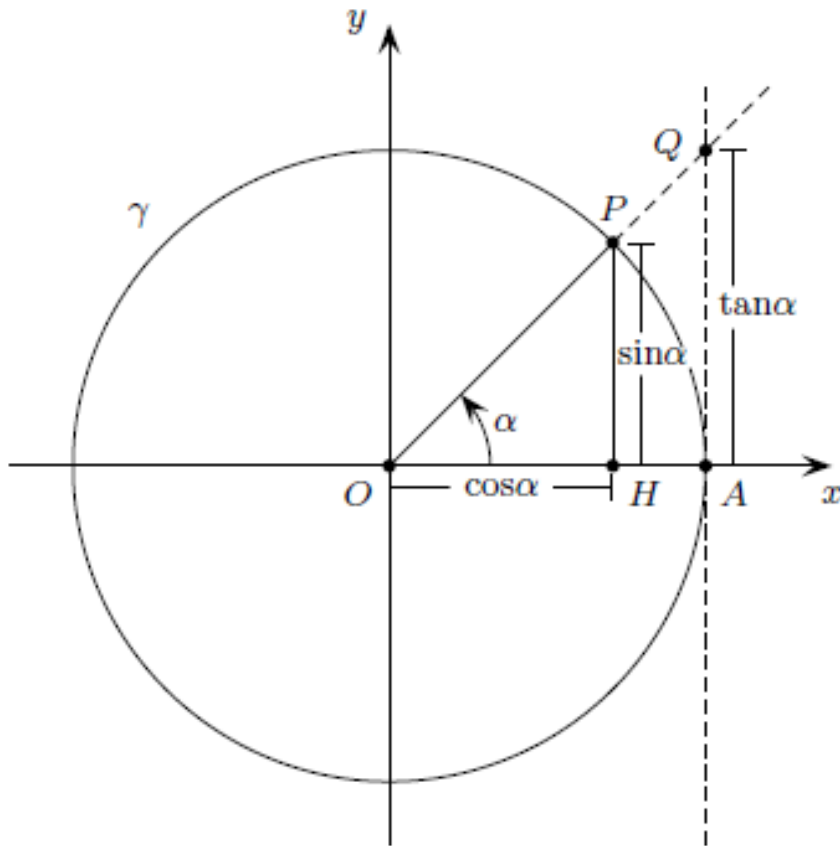


Corso di Laurea in Scienze Naturali

TRIGONOMETRIA

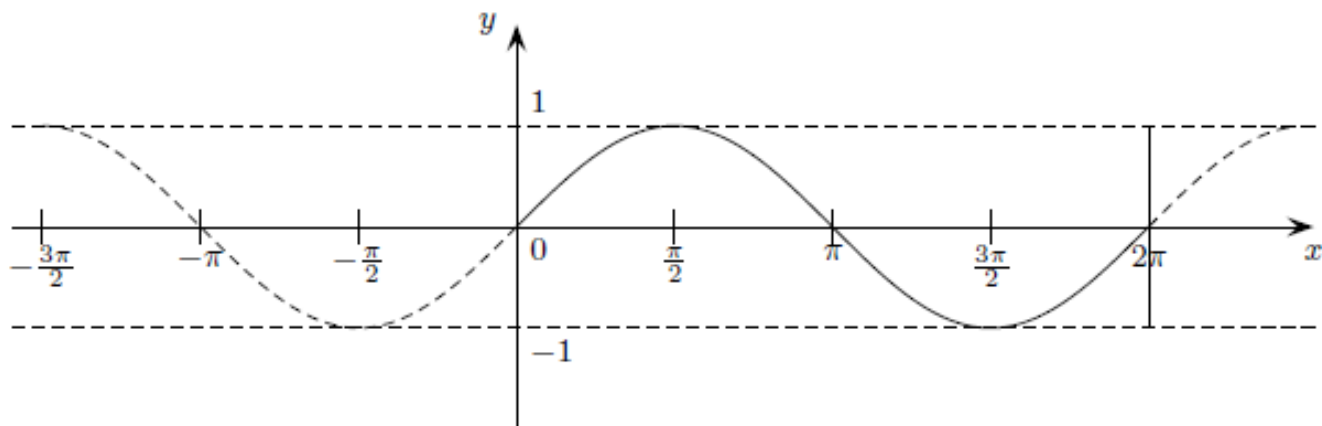
Docente: Marianna Saba

Circonferenza goniometrica

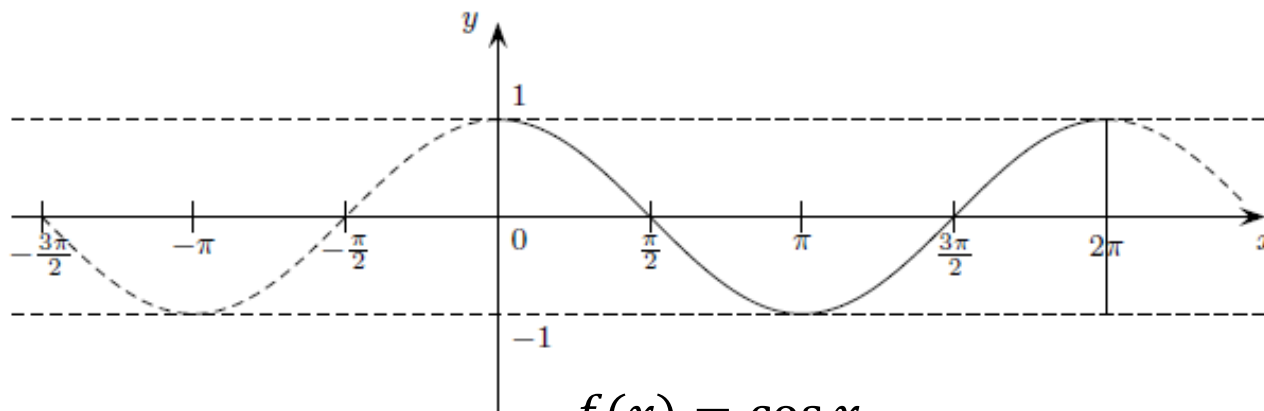


- $A=(1,0)$
- $\alpha = \frac{\text{arco } AP}{AO}$ radianti
 $\frac{\pi}{2} \text{ rad} = 90^\circ$; $\pi \text{ rad} = 180^\circ$;
 $\frac{3\pi}{2} \text{ rad} = 270^\circ$; $2\pi \text{ rad} = 360^\circ$
- α positivo quando P si muove in senso antiorario.
- $\forall \alpha \in [0, 2\pi]$ definiamo
 $\sin \alpha = \text{ordinata di } P$
 $\cos \alpha = \text{ascissa di } P$

Grafici delle funzioni $\sin x$ e $\cos x$



$$f(x) = \sin x$$



$$f(x) = \cos x$$

Proprietà della funzione $\sin x$

- **Funzione periodica di periodo**

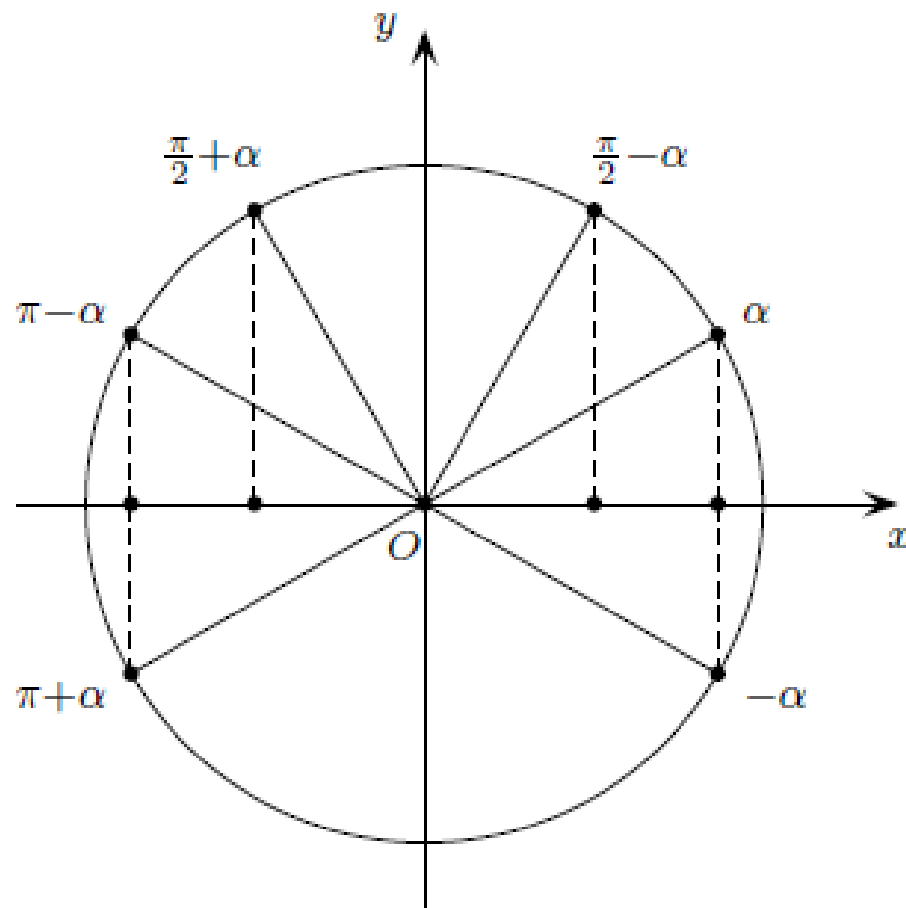
$$T = 2\pi:$$

$$\sin x = \sin(x + T), \forall x \in \mathbb{R}$$

in generale:

$$\sin x = \sin(x + kT), k \in \mathbb{Z}$$

- $\sin(-x) = -\sin x$
($\rightarrow \sin x$ è funzione dispari)
- $\sin(x + \pi) = -\sin x$
- $\sin(\pi - x) = \sin x$



Proprietà della funzione $\cos x$

- **Funzione periodica di periodo**

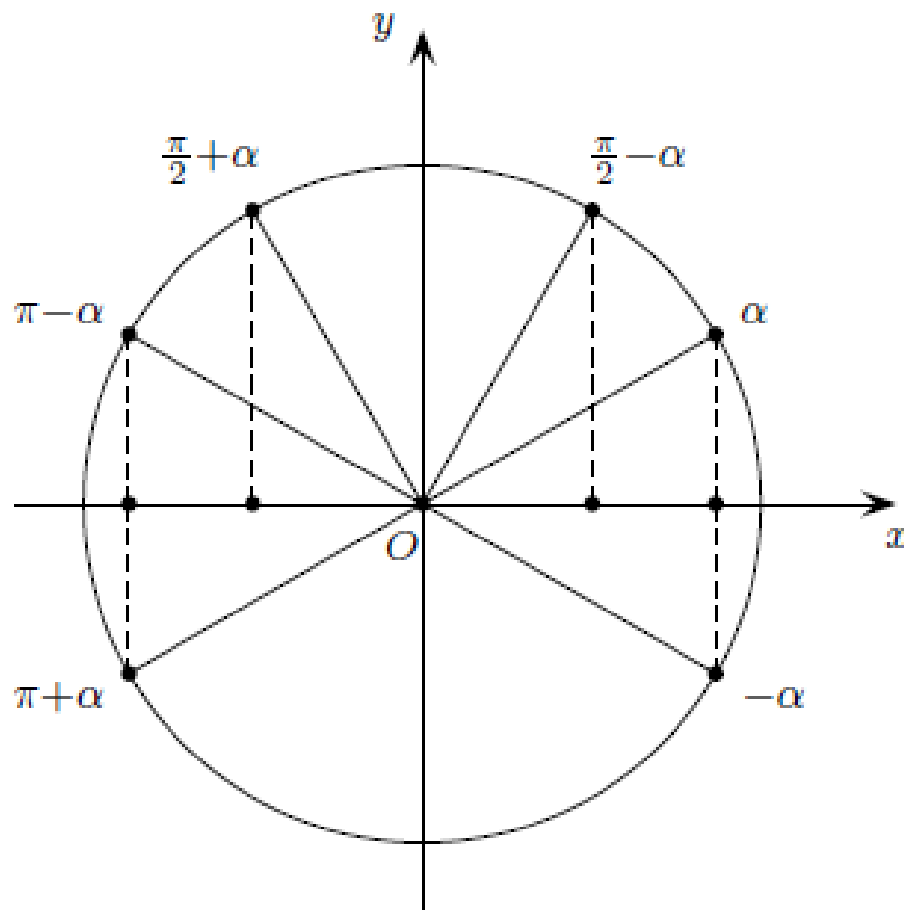
$$T = 2\pi:$$

$$\cos x = \cos(x + T), \forall x \in \mathbb{R}$$

in generale:

$$\cos x = \cos(x + kT), k \in \mathbb{Z}$$

- $\cos x = \cos(-x)$
($\rightarrow \cos x$ è funzione pari)
- $\cos(x + \pi) = -\cos x$
- $\cos(\pi - x) = -\cos x$



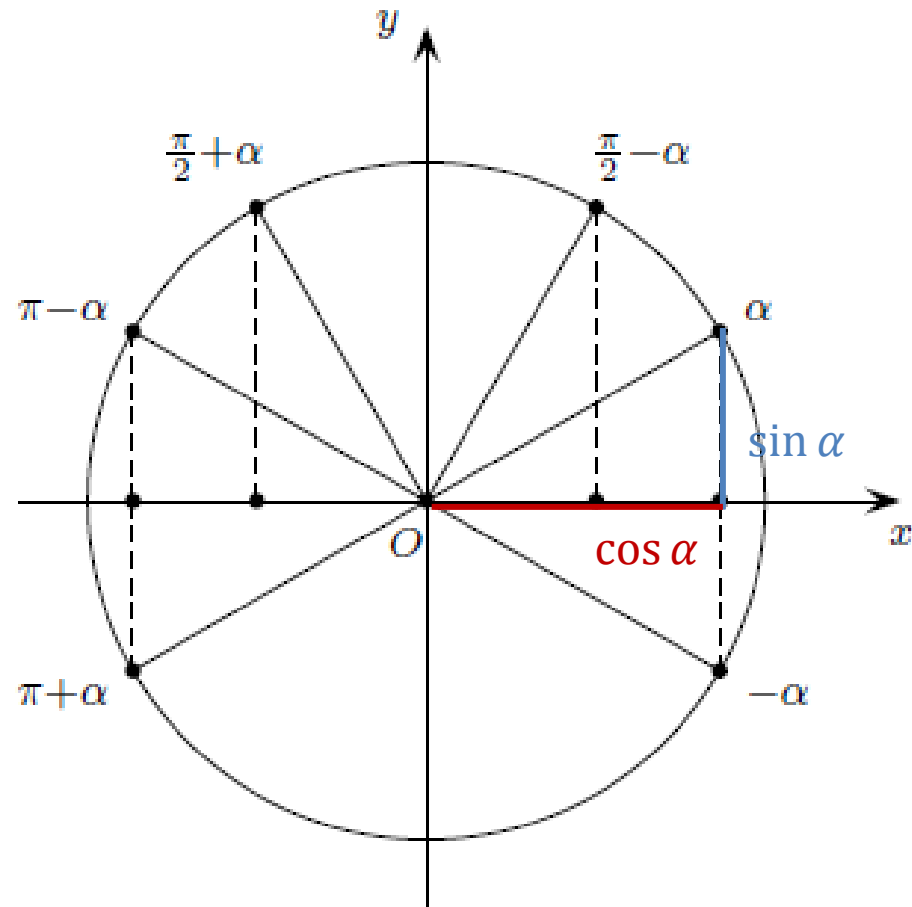
Ulteriori proprietà

- $\sin\left(x + \frac{\pi}{2}\right) = \cos x$
- $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

Relazione fondamentale:

$$\sin^2 x + \cos^2 x = 1$$

(dal Teorema di Pitagora)



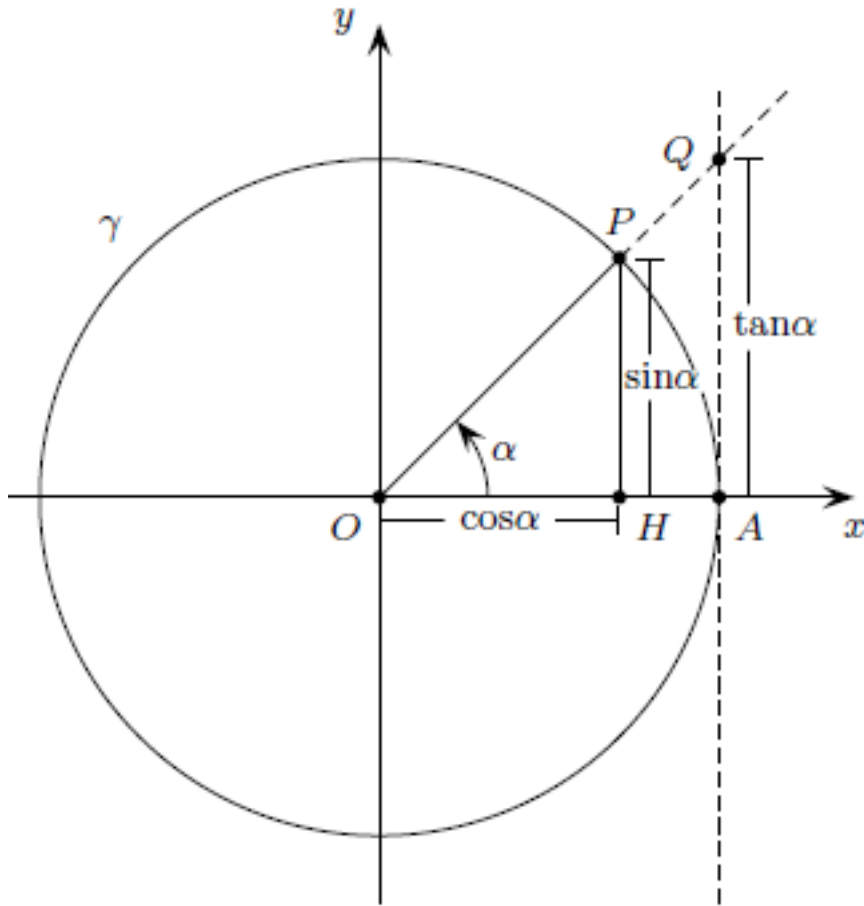
Formule di somma, sottrazione, duplicazione e bisezione per $\sin \alpha$ e $\cos \alpha$

- $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
- $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
- $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
- $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

- $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
 $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

- $\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$ $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$

La funzione $\tan x$



- $\tan x = \frac{\sin x}{\cos x}$

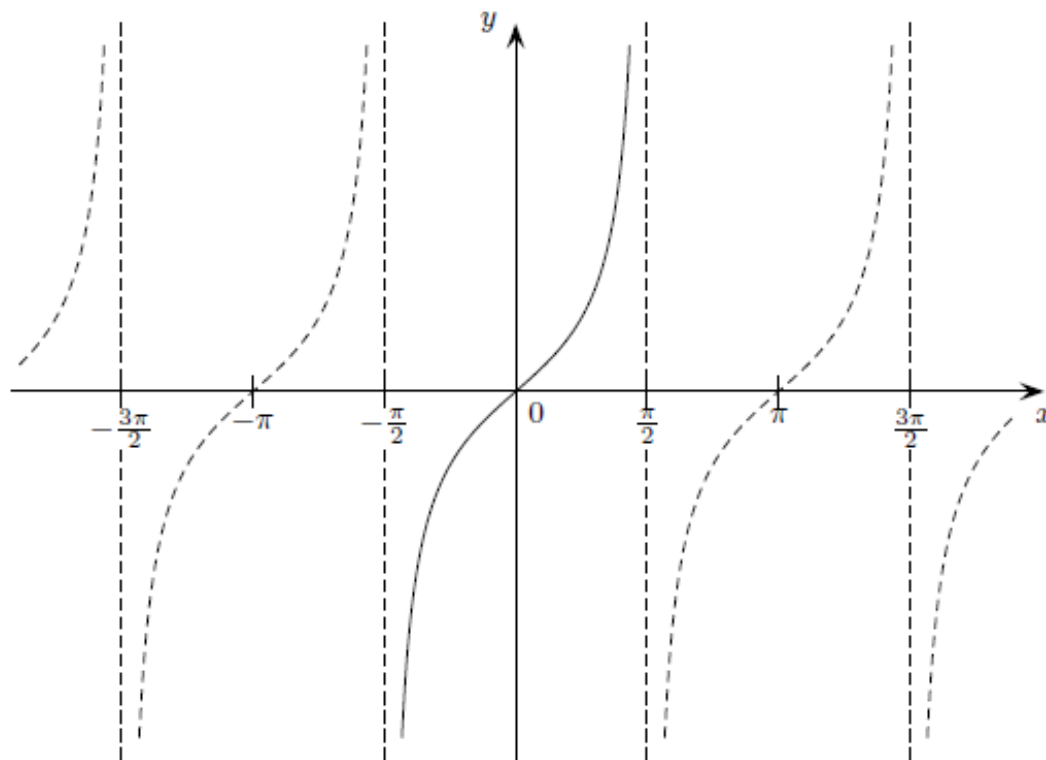
$\forall x \in \mathbb{R}: \cos x \neq 0$, cioè

$\forall x \in \mathbb{R}: x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

- Funzione periodica di periodo
 $T = \pi$

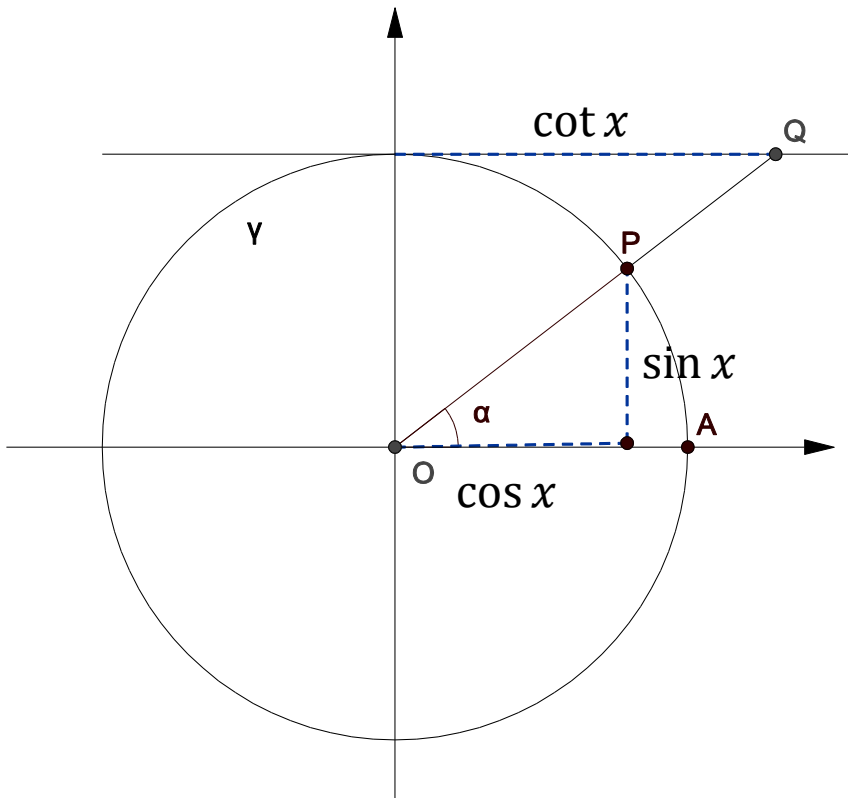
$\tan x = \tan(x + k\pi), k \in \mathbb{Z}$

Grafico della funzione $\tan x$



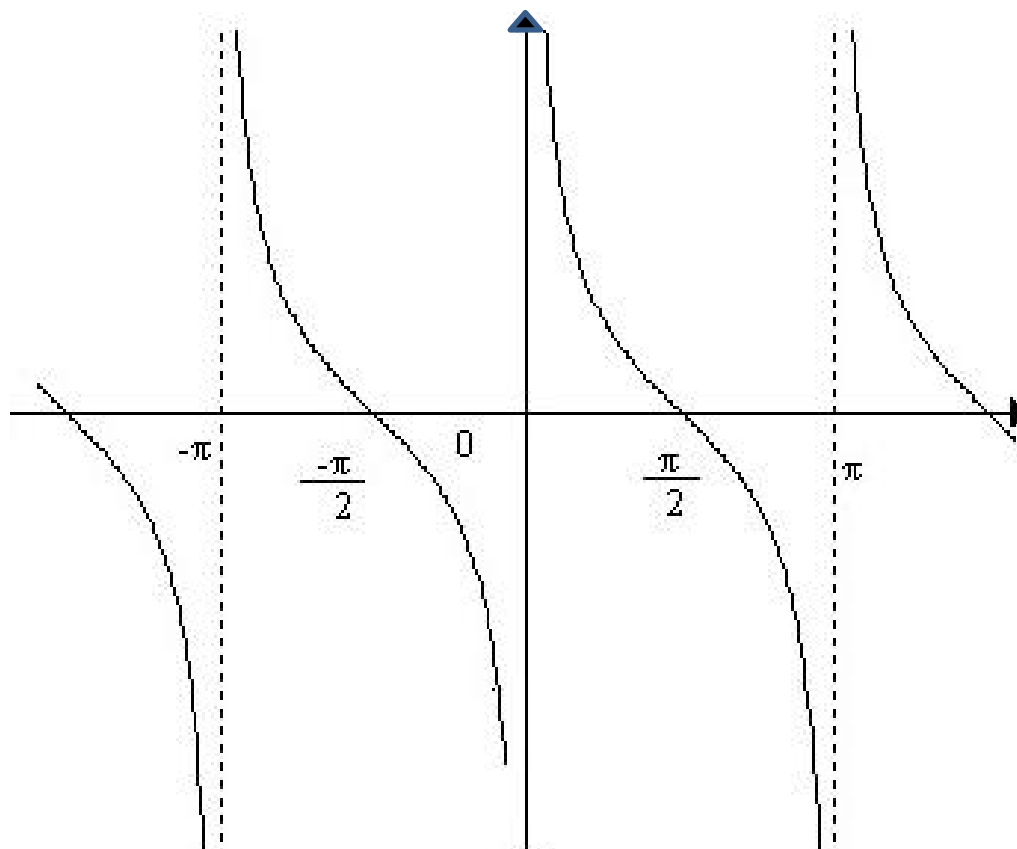
$$f(x) = \tan x$$

La funzione $\cot x$



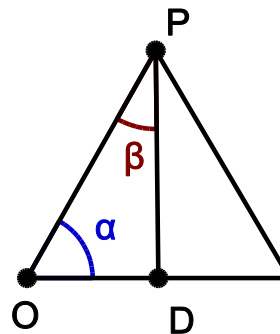
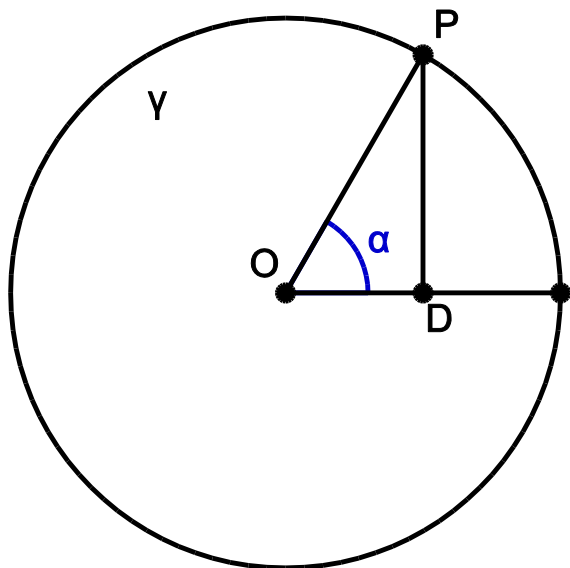
- $\cot x = \frac{\cos x}{\sin x}$
 $\forall x \in R: \sin x \neq 0$, cioè
 $\forall x \in R, x \neq k\pi, k \in Z$
- Funzione periodica di periodo $T = \pi$
 $\cot x = \cot(x + k\pi), k \in Z$
- $\cot x = \frac{1}{\tan x}$

Grafico della funzione $\cot x$



$$f(x) = \cot x$$

Dimostrazione: $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$



$$\alpha = 60^\circ = \frac{\pi}{3} \quad \beta = 30^\circ = \frac{\pi}{6}$$

$$OP = 1$$

$$OD = \frac{1}{2}$$

$$PD = \sin \alpha = \sin \frac{\pi}{3} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$$

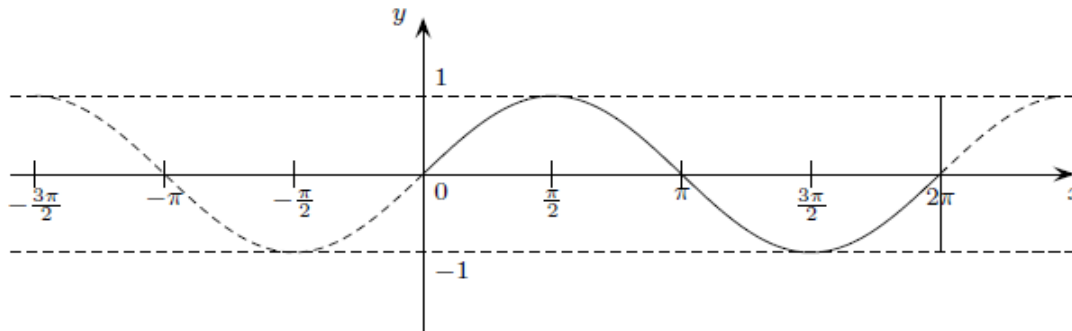
Angoli notevoli

- $\alpha = \frac{\pi}{4}$: $\sin \alpha = \frac{\sqrt{2}}{2}$; $\cos \alpha = \frac{\sqrt{2}}{2}$
- $\alpha = \frac{\pi}{3}$: $\sin \alpha = \frac{\sqrt{3}}{2}$; $\cos \alpha = \frac{1}{2}$
- $\alpha = \frac{\pi}{6}$: $\sin \alpha = \frac{1}{2}$; $\cos \alpha = \frac{\sqrt{3}}{2}$

Calcolare $\tan \frac{\pi}{6}$, $\cot \frac{\pi}{3}$, $\tan \frac{\pi}{4}$

Funzioni inverse: $f^{-1}(x) = \arcsin x$

- $f: \mathbb{R} \rightarrow \mathbb{R}$ definita da $f(x) = \sin x$

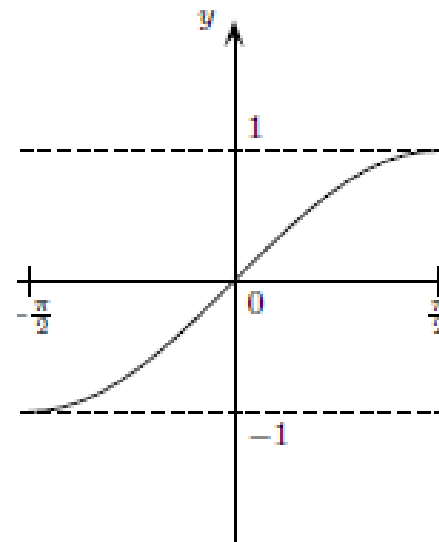


NON è
invertibile

- E' invertibile se definisco

$$f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$$

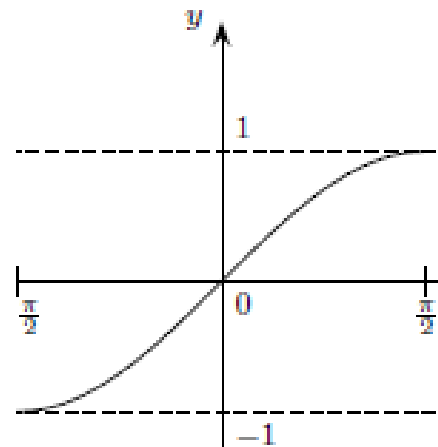
$$f(x) = \sin x$$



La funzione arcsin(x)

$$f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$$

$$f(x) = \sin x$$

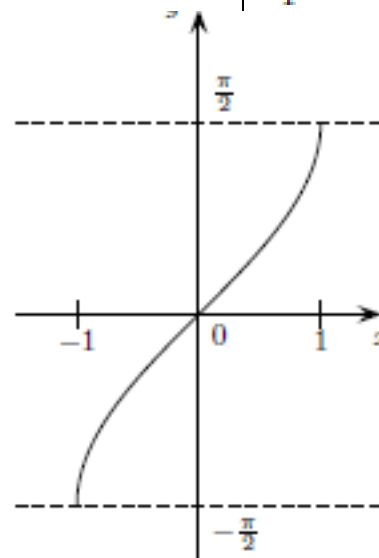


- L'inversa è

$$f^{-1}: [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$f^{-1}(x) = \arcsin(x)$$

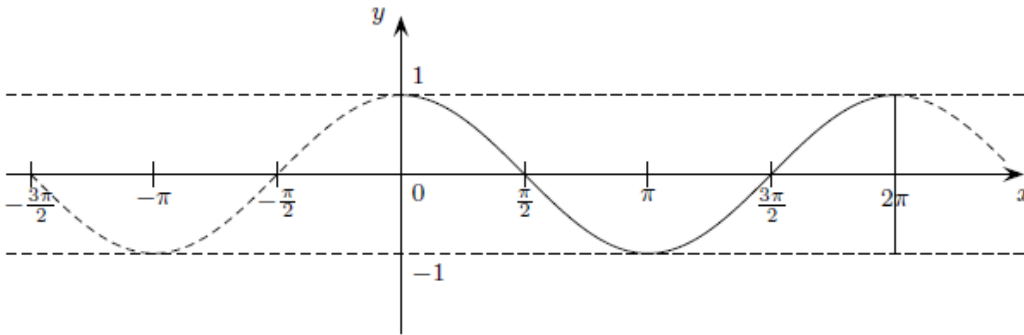
- f^{-1} è dispari



Es: $\arcsin(-1) = -\frac{\pi}{2}$; $\arcsin(0) = 0$; $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Funzioni inverse: $f^{-1}(x) = \arccos x$

- $f: \mathbb{R} \rightarrow \mathbb{R}$ definita da $f(x) = \cos x$

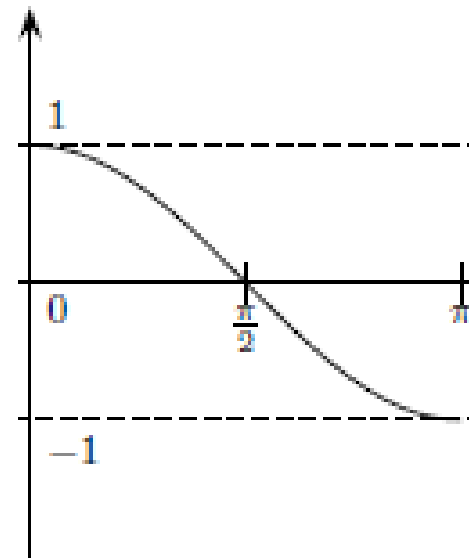


NON è
invertibile

- E' invertibile se definisco

$$f: [0, \pi] \rightarrow [-1, 1]$$

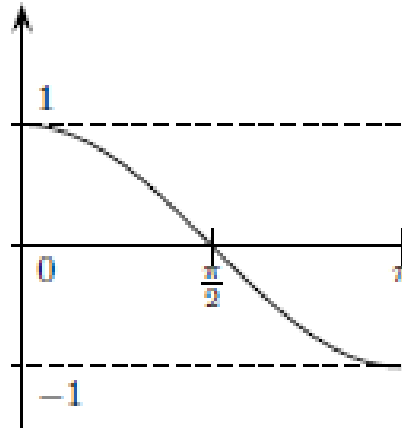
$$f(x) = \cos x$$



La funzione $\arccos(x)$

$$f: [0, \pi] \rightarrow [-1, 1]$$

$$f(x) = \cos x$$

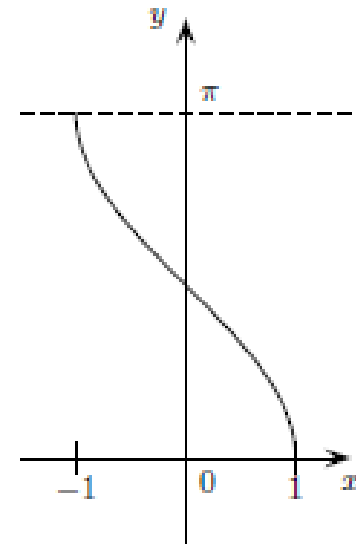


- L'inversa è

$$f^{-1}: [-1, 1] \rightarrow [0, \pi]$$

$$f^{-1}(x) = \arccos(x)$$

- f^{-1} è dispari

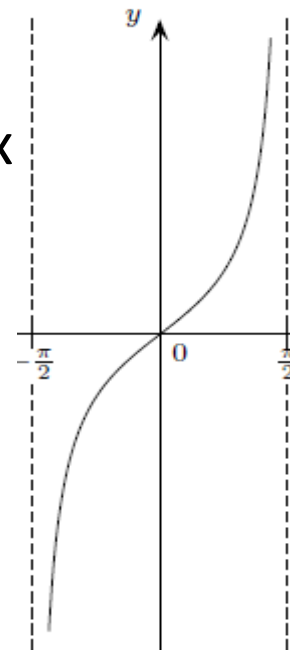


$$\text{Es: } \arccos(-1) = \pi; \quad \arccos(0) = \frac{\pi}{2}; \quad \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

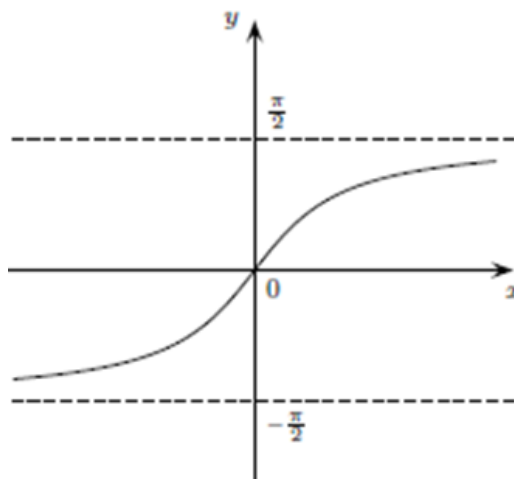
La funzione $\arctan(x)$

- $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow R$
 $f(x) = \tan x$

$$f(x) = \tan x$$



- L'inversa è
 $f^{-1}: R \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$
 $f^{-1}(x) = \arctan x$
- f^{-1} è dispari

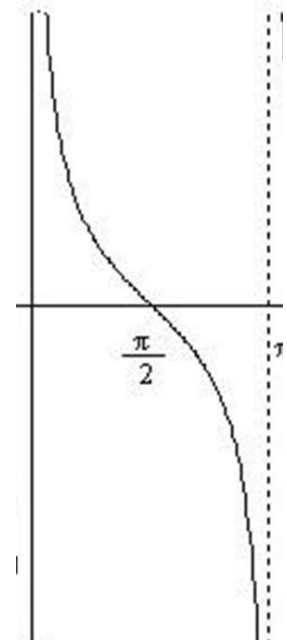


- Es: $\arctan(0) = 0$ $\arctan \sqrt{3} = \frac{\pi}{3}$ $\arctan(1) = \frac{\pi}{4}$

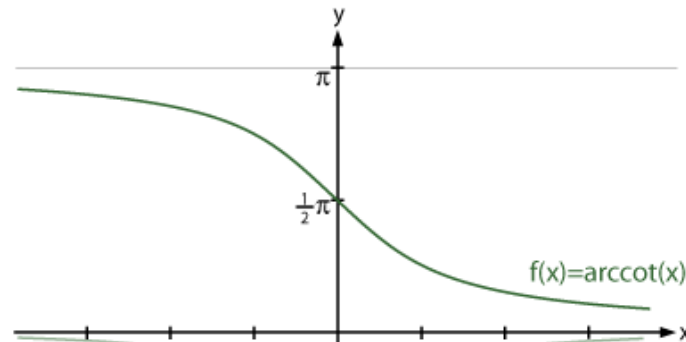
La funzione $\operatorname{arccot}(x)$

- $f: (0, \pi) \rightarrow \mathbb{R}$
 $f(x) = \cot x$

$$f(x) = \cot x$$



- L'inversa è
 $f^{-1}: \mathbb{R} \rightarrow (0, \pi)$
 $f^{-1}(x) = \operatorname{arccot}(x)$



$$\text{Es: } \operatorname{arccot} 1 = \frac{\pi}{4}$$

Equazioni

- $\sin x = -2 \quad x=?$

$$\nexists x \in R$$

- $\sin x = \frac{1}{2} \quad x=?$

$$x = \pi - \frac{\pi}{6} + 2k\pi$$

$$x = \frac{\pi}{6} + 2k\pi \quad (k \in Z)$$

- $\sin 2x = \frac{\sqrt{2}}{2} \quad x=?$

$$2x = \frac{\pi}{4} + 2k\pi$$

$$2x = \pi - \frac{\pi}{4} + 2k\pi$$

quindi $x=.....$

Equazioni

- $\sin 5x = \sin 3x$

$$5x = 3x + 2k\pi$$

$$5x = \pi - 3x + 2k\pi$$

quindi $x = k\pi$ e $x = \frac{\pi}{8} + \frac{k\pi}{4}$

- $\sin^2 x + \sin x - 2 = 0$

Posto $\sin x = t$, si ha $t^2 + t - 2 = 0$

con soluzioni $t = -2$, $t = 1$ cioè

$\sin x = -2$ e $\sin x = 1$, ovvero

$$x = \frac{\pi}{2} + 2k\pi.$$

Equazioni

- $\cos x = \frac{\sqrt{3}}{2} \quad x=?$

$$x = \frac{\pi}{6} + 2k\pi$$

$$x = -\frac{\pi}{6} + 2k\pi \quad (k \in \mathbb{Z})$$

- $\cos 7x = \cos \frac{x}{3} \quad x=?$

$$7x = \frac{x}{3} + 2k\pi$$

$$7x = -\frac{x}{3} + 2k\pi$$

- $\sqrt{2} \sin x \cos x + \cos x = 0$

$$\cos x (\sqrt{2} \sin x + 1) = 0$$

$$x = \frac{\pi}{2} + k\pi \quad x = \frac{5\pi}{4} + 2k\pi \quad x = \frac{7\pi}{4} + 2k\pi$$

Equazioni

- $\tan 4x = -1$

$$4x = \frac{3}{4}\pi + k\pi \rightarrow x = \frac{3}{16}\pi + k\frac{\pi}{4}$$

- $\sin^2 x - 3 \cos^2 x = 0$

Usando la Formula fondamentale della trigonometria si ha

$$1 - \cos^2 x - 3 \cos^2 x = 0$$

$$x = \pm \frac{\pi}{3} + k\pi$$

Disequazioni

- $\sin x > -\frac{1}{2}$

Si risolve prima $\sin x = -\frac{1}{2}$

$$x = \frac{7\pi}{6} + 2k\pi \quad x = -\frac{\pi}{6} + 2k\pi$$

- Soluzione:

$$-\frac{\pi}{6} + 2k\pi < x < \frac{7\pi}{6} + 2k\pi$$

Disequazioni

- $-\cos x - \frac{\sqrt{2}}{2} \geq 0$

Si risolve prima $\cos x + \frac{\sqrt{2}}{2} = 0$

$$x = \frac{3\pi}{4} + 2k\pi \quad x = \frac{5\pi}{4} + 2k\pi$$

- Soluzione:

$$\frac{3\pi}{4} + 2k\pi \leq x \leq \frac{5\pi}{4} + 2k\pi$$

Disequazioni

- $\sqrt{3} \sin x - \cos x > 0$

Dividiamo per $\cos x \neq 0$, $\left(x \neq \frac{\pi}{2} + k\pi\right)$ e distinguiamo:

1. $\cos x > 0$

2. $\cos x < 0$

$$1. \begin{cases} \frac{\sqrt{3} \sin x - \cos x}{\cos x} > 0 \\ \cos x > 0 \end{cases}$$

$$2. \begin{cases} \frac{\sqrt{3} \sin x - \cos x}{\cos x} < 0 \\ \cos x < 0 \end{cases}$$

cioè

$$1. \begin{cases} \sqrt{3} \tan x - 1 > 0 \\ \cos x > 0 \end{cases}$$

$$2. \begin{cases} \sqrt{3} \tan x - 1 < 0 \\ \cos x < 0 \end{cases}$$

1. Soluzione del primo sistema

$$\frac{\pi}{6} + 2k\pi < x < \frac{\pi}{2} + 2k\pi$$

2. Soluzione del secondo sistema

$$\frac{\pi}{2} + 2k\pi < x < \frac{7\pi}{6} + 2k\pi$$

- Poiché anche $x = \frac{\pi}{2} + 2k\pi$ è soluzione, facendo l'unione si ha:

$$\frac{\pi}{6} + 2k\pi < x < \frac{7\pi}{6} + 2k\pi$$