Spaces of foliations on surfaces

ATHANASE PAPADOPOULOS

Résumé. On définit certains espaces de classes d’équivalence de feuilletages sur les surfaces, comme limites inductives d’espaces de poids sur des réseaux ferroviaires. Dans le cas particulier de l’espace des feuilletages affines, la topologie naturelle sur la limite inductive coïncide avec celle qui a été définie par Hatcher et Oertel.

1. INTRODUCTION

In this paper, we consider foliations on surfaces, whose singularities are isolated and which are either centers or $n$-prongs, with $n$ any integer $\geq 1$ and $\neq 2$. These types of singularities are described in Figure 1. We shall call sometimes such a foliation a foliation with general type singularities, in order to distinguish it from foliations with a more restricted type of singularities which we shall also consider below.

![Fig. 1. A center and $n$-prongs, with $n = 1, 2, 3$.](image)

We are interested in foliations defined up to isotopy and Whitehead moves. We recall that a Whitehead move is a transformation of the foliated surface, which consists either in collapsing to a point a compact leaf joining two singularities, or in the inverse operation. An example of a Whitehead move is given in Figure 2. The other Whitehead moves involve other types of singular points. We let $\mathcal{F}$ denote the set of equivalence classes of foliations with general type singularities.

(*) Institut de Mathématiques, CNRS and Universit\'e Louis Pasteur, 7 rue Ren\'e Descartes, 67084 Strasbourg Cedex, France. Presentato l’8/04/03
The foliations considered here are more general than those studied in [1], [5] and [8], where the singular points are only allowed to be \( n \)-prongs with \( n \geq 3 \) and which have no centers. As we shall see below, the fact of allowing singularities of general type makes some of the techniques more efficient. For instance, in this setting, one can take sums and convex combinations of weighted train tracks. An advantage of this general setting is that we can define the topology of the space \( \mathcal{PAF} \) of projective classes of affine foliations, introduced by Hatcher and Oertel in [2], without having recourse to closed curves in the universal abelian cover of \( S \). Indeed, this can be done by defining first the topology of the space \( \mathcal{PAF} \) of projective classes of affine foliations with general type singularities, by considering that space as an inductive limit of spaces of foliations carried by train tracks, and then restricting the topology to the subspace \( \mathcal{PAF} \).

We mention that oriented affine foliations with general type singularities have been studied by I. Lioussse in [3]. We shall study the space of such foliations here, although from a different point of view (we consider equivalence classes of foliations).

Let us mention a few spaces of foliations which are of particular interest and which are related to the space \( \mathcal{F} \). These are the space \( \mathcal{AF} \) of equivalence classes of affine foliations with general type singularities, the space \( \mathcal{AF} \) of equivalence classes of affine foliations with \( n \)-prong singularities with \( n \geq 3 \) and no centers, the subspace \( \mathcal{OAF} \subset \mathcal{AF} \) of equivalence classes of orientable affine foliations with general type singularities, and the subset \( \mathcal{OAF} \subset \mathcal{OAF} \) of equivalence classes of orientable affine foliations with \( n \)-prong singularities, \( n \geq 3 \). We shall define the topologies of these various spaces and discuss some of their properties.

Let \( S \) be an oriented surface (compact or not) without boundary.

**Definition 1.1 (Train Track).** A train track \( \tau \) on \( S \) is a graph with trivalent vertices which is embedded in \( S \) and which is «smooth» at each vertex, as represented in Figure 3.

We note that no differentiable structure is needed to define this smoothness; it suffices to have at each vertex of \( \tau \) a notion of two sides, one side at which two edges abut and one side at which one edge abuts. The vertices of \( \tau \) are called switches, and the piece of surface which is near a switch and which is contained between the two edges abutting from the same side at a switch, is called a spike. Each connected component of \( S - \tau \) is thus a surface with boundary which has a certain number of spikes.
A sub-train track of \( \tau \) is a train track \( \tau' \) which is smoothly embedded in \( \tau \). Here, the term \textit{smoothly} means that whenever the interior of an edge of \( \tau' \) contains a switch of \( \tau \), then, at this point, the edge of \( \tau' \) crosses completely the switch of \( \tau \) from one side to the other side. A sub-train track is naturally a train track.

A fibered neighborhood of \( \tau \) is a tubular neighborhood which is equipped with a foliation by segments which are called the \textit{ties}. The \textit{singular ties} are those which contain the switches. The model near a singular tie is given in Figure 4. The fibered neighborhood of \( \tau \) is well-defined up to isotopy, and the train track \( \tau \), up to isotopy, is the quotient of its fibered neighborhood by a map which collapses each tie to a point.

A partial foliation \( F^* \) on \( S \) is a foliation whose support is a subsurface with boundary of \( S \), with \textit{internal spikes} on the boundary of this support (an internal spike being a local piece of surface which is a complementary region of a spike considered after Definition 1.1). From a partial foliation \( F^* \), we can obtain a genuine foliation by collapsing each connected component of \( S - F^* \) onto a \textit{spine}, that is, a zero- or a one-dimensional complex having the homotopy type of that component, taking into account the smoothness (see Figure 5), creating singular points of the types described in Figure 1. In the case where the connected component is a disk with no spike on its boundary, we can take a spine to be a point, and the singular point that is created is a center. In all the other cases, the spine
is one-dimensional. A spine is well-defined up to Whitehead moves of one-dimensional complexes. Thus, collapsing each component of $S - F^*$ onto a spine, we obtain a measured foliation $F$ on $S$ which is well-defined up to isotopy and Whitehead moves of the foliation.

**Definition 1.2 (Foliation carried by a train track).** We say that a foliation $F$ on $S$ is carried by a train track $\tau$ if there exists a partial foliation $F^*$ on $S$ whose support is contained in a fibered neighborhood $V(\tau)$ of $\tau$, with the leaves of $F^*$ transverse to the ties and such that, up to isotopy, $F$ is obtained from $F^*$ by collapsing each component of $S - F^*$ onto a spine.

**Proposition 1.3.** Every foliation on $S$ is carried by some train track.

**Proof.** Let $F$ be a foliation on $S$. We blow up each singular point of $F$ which is a center, creating a small nonfoliated disk, and we open up each $n$-prong singularity, pushing the leaves away from this singularity and creating a disk with $n$ spikes. These operations are indicated in Figure 6. We obtain in this way a partial foliation on $S$ which we call $F'$. We cover now the support of $F'$ with rectangles (called «flow boxes») on which the foliation induced by $F'$ is a standard foliation (whose leaves we call the «horizontal leaves»). By compactness of the surface, only a finite number of such flow boxes is needed. In Figure 7, we have represented a flow box with a spike on its boundary. Next, we «unzip» each of the horizontal sides of the flow boxes which are not contained in the horizontal sides of $F'$, creating digons, as indicated in Figure 8.

Let $F''$ be the resulting partial foliation. The support of $F''$ admits a foliation $G$ by segments which are transverse to the leaves of $F''$ and whose endpoints are on the boundary of this support. In fact, it is easy to construct $G$ step by step as a vertical foliation in each flow box. At the spikes, the local model for the leaves of $G$ is given in Figure 9.

Now the quotient space of the support of $G$ by the operation of collapsing each leaf
of $G$ to a point is naturally a train track $\tau$, and the support of $G$, equipped with its vertical foliation, constitutes a fibered neighborhood of $\tau$, which naturally carries the foliation $F$. This proves Proposition 1.3.

\section*{2. AFFINE FOLIATIONS}

We need to introduce a few notions which are slightly more general than the notions that are used in Thurston's theory on surfaces. We start with measured foliations. We shall use them in the definition of general affine foliations.

\begin{definition} \textbf{(Measured Foliations)}. A \textit{measured foliation} $F$ on $S$ is a foliation equipped with an \textit{invariant transverse measure}. This means that each arc in $S$ which is transverse to $F$ is endowed with a measure which is equivalent to the Lebesgue measure of an interval.
\end{definition}

Fig. 6. Blowing up a singularity and creating a non-foliated region.

Fig. 7. A flow box, near a spike.

Fig. 8. Unzipping the horizontal sides of a flow box.
of \( \mathbb{R} \) and such that if a tranverse arc \( s \) is moved to a transverse arc \( s' \) by an isotopy during which each point of \( s \) stays on the same leaf, then the map between \( s \) and \( s' \) which results from this move and which is called a \textit{local holonomy map} is measure preserving.

We consider the equivalence relation on the set of measured foliations which is generated by isotopy and Whitehead moves which preserve the transverse measures of the foliations. We denote by \( \mathcal{MF} \) the set of equivalence classes. We note that this equivalence relation is stronger than the one induced by the relation \( \sim \) which we defined on the set of all foliations on \( S \) (because of the condition on the transverse measure).

**Definition 2.2 (Invariant Measure on a Train Track).** An \textit{invariant measure} on a train track \( \tau \) is the assignment of a nonnegative real number to each edge of \( \tau \), which is called the \textit{weight} of that edge, such that the weights are not all equal to zero and satisfy, at each switch of \( \tau \), the following equation (called the \textit{equation of conservation of mass}): the sum of the weights of the two edges abutting from one side is equal to the weight of the edge abutting from the other side.

From an invariant measure on a train track \( \tau \), we obtain a measured foliation on \( S \), which is well-defined up to isotopy and Whitehead moves and which is carried by \( \tau \), by the following well-known construction.

**Construction 2.3 (Measured Foliation from Invariant Measure).** Consider a fibered neighborhood \( V(\tau) \) of \( \tau \), equipped with its foliation by the ties. For each edge \( e \) of \( \tau \) whose weight is non-zero, consider a thin rectangle in \( V(\tau) \) which has two of its opposite sides (the «vertical sides») contained in the ties and the other two sides (the «horizontal sides») transverse to the ties, with the image of this rectangle by the natural projection \( V(\tau) \to \tau \) being equal to the edge \( e \). We endow this rectangle with a standard foliation, whose leaves are segments transverse to the ties, and we equip this foliation with a transverse measure for which the total measure of a vertical side is equal to the weight of the edge above which the rectangle lies. The rectangles are then glued together along vertical sides which lie above the same switches by measure-preserving maps. In this way, we obtain a measured partial foliation \( F^* \) on \( S \) which is equipped with an invariant transverse measure. Collapsing each connected component of \( S - F^* \) onto a spine, we obtain a measured foliation which is well-defined up to isotopy and Whitehead moves.
Conversely, any measured foliation $F$ can be obtained using this construction, starting with a train track equipped with an invariant measure. To see this, we use the train track $\tau$ and the foliation $G$ (which is transverse to $F$), described in the proof of Proposition 1.3. Each leaf of the foliation $G$ is naturally equipped with a measure (which comes from the transverse measure of $F$), and the various leaves of $G$ which lie above the same edge of $\tau$ have the same total measure. This induces a natural system of weights on the edges of $\tau$, which is an invariant measure, and $F$ can be recovered from this invariant measure by Construction 2.3.

In the rest of this paper, $S$ is an oriented closed surface of genus $g \geq 2$, and $\tilde{S} \to S$ is its universal covering.

**Definition 2.4 (Affine Foliation).** An affine foliation $F$ on $S$ is a foliation satisfying the following two properties:

1. (2.4.1) The lift $\tilde{F}$ of $F$ to $\tilde{S}$ is equipped with an invariant transverse measure.
2. (2.4.2) For each covering translation $g$ of $\tilde{S}$, there exists a positive real number $q(g)$ such that if $\mu$ denotes the transverse measure of $\tilde{F}$, then for each arc $s$ in $\tilde{S}$ which is transverse to $\tilde{F}$, we have $\mu(g(s)) = q(g)\mu(s)$.

Let $\Gamma$ be the group of covering translations of the covering $\tilde{S} \to S$. The map $\Gamma \to \mathbb{R}_+^*$ defined by $g \to q(g)$ is a homomorphism. Thus, using the canonical isomorphism $\pi_1(S) \cong \Gamma$, we obtain a well-defined homomorphism $\pi_1(S) \to \mathbb{R}_+^*$, which is called the holonomy homomorphism of $F$. We shall denote this homomorphism by the same letter $q$.

There is an equivalence relation on the set of affine foliations, generated by isotopy and Whitehead moves performed on the foliation $F$ on $S$ (we are using the notations of Definition 2.4), such that the lift of these moves to the universal cover $\tilde{S}$ preserve the transverse measure of the lifted foliation $\tilde{F}$. This equivalence relation between affine foliations is stronger than the equivalence relation induced by the relation $\sim$ on the space of foliations on $S$. We denote by $\mathcal{AF} = \mathcal{AF}(S)$ the set of equivalence classes.

We let $\mathcal{AFF}_q \subset \mathcal{AF}$ be the set of affine foliations with holonomy homomorphism $q$. The group $\mathbb{R}_+^*$ of positive reals acts naturally on the spaces $\overline{\mathcal{AF}}$ and $\overline{\mathcal{AFF}_q}$ (by multiplying the transverse measure of a representative by a constant factor), and we let $\overline{PAF}_q$ and $\overline{PAF}_q$ denote respectively the quotient spaces.

We shall parametrize spaces of affine foliations by using pairs of transverse train tracks (one of which is oriented), and for this we introduce the following.

An oriented train track (respectively a transversely oriented train track) is a train track equipped with an orientation (respectively a transverse orientation) on each of its edges, such that the orientations (respectively the transverse orientations) are coherent at the vertices, as in Figure 10.
A transversely oriented train track $\sigma$ equipped with an invariant measure defines an element $\varphi \in \text{Hom}(\pi_1(S), \mathbb{R}^*_+) $, in the following way: For $g \in \pi_1(S)$, we take an oriented loop $\gamma$ representing $g$, intersecting $\sigma$ transversely and such that no intersection point of $\gamma$ with $\sigma$ is a switch. Let $a_1, \ldots, a_n$ be the intersection points of $\gamma$ with the edges of $\sigma$ which have positive weights. Without loss of generality, we can assume that there exists at least one such point. For each $i = 1, \ldots, n$, let $w_i$ be the weight (respectively the opposite of the weight) of the edge containing $a_i$, if the orientation of $\gamma$ agrees (respectively disagrees) with the transverse orientation of $\sigma$ at that point. We then define $\varphi(g) = \prod_{i=1}^n \exp^{w_i}$. It is easy to see that this value is independent of the choice of the curve $\gamma$ representing $g$. This defines the homomorphism $\varphi$.

By the canonical isomorphism $\text{Hom}(\pi_1(S), \mathbb{R}^*_+) \cong H^1(S; \mathbb{R})$, the train track $\sigma$, equipped with its transverse orientation and its invariant measure, defines an element of $H^1(S; \mathbb{R})$.

The map which assigns to an invariant measure on a fixed oriented train track $\sigma$ its class in $H^1(S; \mathbb{R})$ is not injective in general.

The following definition and construction are slightly more general than a definition and a proposition in [4] restricted to the case of surfaces.

**Definition 2.5 (Broken Measure).** Let $\sigma$ be a transversely oriented train track equipped with an invariant measure and let $\tau$ be a train track which is transverse to $\sigma$. We suppose that no intersection point of $\tau$ with $\sigma$ is a switch of $\tau$ or of $\sigma$. We call a connected components of $\tau - (\{\text{switches of } \tau\} \cup \sigma)$ an edge of $(\tau, \sigma)$. A broken measure on the pair $(\tau, \sigma)$ is then a collection of nonnegative numbers (called weights) associated to the edges of $(\tau, \sigma)$, satisfying the following conditions:

(2.5.1) The weights are not all zero.

(2.5.2) At each switch of $\tau$, the equation of conservation of mass is satisfied.

(2.5.3) At each intersection point of $\tau$ with an edge of $\sigma$ whose weight is $w$, the weight on the edge of $(\tau, \sigma)$ which is after the intersection point (with respect to the transverse orientation of $\sigma$) is equal to $\exp^w \times$ the weight of the edge which is before that intersection point.

**Construction 2.6. (Affine foliation from broken measure).** Let $\varphi \in H^1(S; \mathbb{R})$ be the cohomology class defined by $\sigma$ equipped with its tranverse orientation and invariant measure. From a broken measure on $(\varphi, \sigma)$, we obtain a well-defined element of $\overline{\mathcal{PAF}}_{\varphi}$, Fig. 10. An oriented and a transversely oriented train track.
in the following way. We first define a foliation $F$ on $S$. Let $V(\tau)$ be a fibered neighborhood of $\tau$. We can assume without loss of generality that the weights on the edges of $\sigma$ are all nonzero (otherwise, we replace $\sigma$ by a sub-train track) and that the intersection of $\sigma$ with the support of $F^*$ is a union of segments contained in the ties of $V(\tau)$. We consider a partial foliation $F^*$ whose support is contained in $V(\tau)$, obtained as in Construction 2.3 above, except for the transverse measure. Thus, for each edge $e$ of $(\tau, \sigma)$, we consider a foliated rectangle whose support is contained in $V(\tau)$, having two of its opposite sides (the «vertical» sides) in the ties and the other two sides (the «horizontal» sides) transverse to the ties, and such that the image of this rectangle by the natural projection $V(\tau) \to \tau$ is the edge $e$. We foliate the rectangle by segments which are transverse to the ties, and we equip this («horizontal») foliation with an invariant transverse measure for which the measure of a vertical side is equal to the weight of the edge $e$. We glue together along their vertical sides the rectangles which lie above the same switches of $\tau$ by measure preserving maps, and along the sides which are above points of $\tau \cap \sigma$ by affine maps. Let $F^*$ be the resulting partial foliation on $S$. The connected components of $F^* - \sigma$ are equipped with measured foliations. Let $\tilde{F}^*$ and $\tilde{\sigma}$ be respectively the lifts of $F^*$ and $\sigma$ to the universal cover $\tilde{S}$ of $S$. The intersection of $\tilde{\sigma}$ with the support of $\tilde{F}^*$ is a union of segments with endpoints on the boundary of this support, and each connected component of $\tilde{F}^* - \tilde{\sigma}$ is equipped with a measured foliation. We modify the transverse measure in each of these components by multiplying it by an appropriate constant, so that the result is a partial measured foliation on the surface $\tilde{S}$. For that purpose, we begin by choosing a connected component $S_0$ of $\tilde{S} - \tilde{\sigma}$ and in this component we keep the transverse measure of the foliation induced by $\tilde{F}^*$ unchanged. (Note that this induced foliation may be the empty foliation.) We modify then the transverse measure of the other foliated connected components of $\tilde{S} - \tilde{\sigma}$ as follows. Let $S_1$ be such a component. We choose a path $s_{0,1}$ from a point $p_0$ in the interior of $S_0$ to a point $p_1$ in the interior of $S_1$, which is in general position with respect to $\sigma$. It is convenient to assume that $s_{0,1}$ has nonempty intersection with $\tilde{\sigma}$, and we can make this assumption without loss of generality. We orient $s_{0,1}$ from $p_0$ to $p_1$, and we let $a_0, \ldots, a_n$ be the sequence of intersection points of $s_{0,1}$ with $\tilde{\sigma}$. For each $i = 1, \ldots, n$, let $w_i$ be the weight (respectively the opposite of the weight) of the edge of $\tilde{\sigma}$ at $a_i$ if the orientation of $s_{0,1}$ agrees (respectively disagrees) with the transverse orientation of $\tilde{\sigma}$ at this point. We multiply then the transverse measure of the foliation induced by $\tilde{F}^*$ on the component $S_1$ by the factor $\prod_{i=1}^n \exp w_i$. It is easy to see that the definition of this factor does not depend on the choice of the path $s_{0,1}$ joining $p_0$ to $p_1$ and that the transverse measures that we obtain in this way on the various components of $\tilde{F}^* - \tilde{\sigma}$ agree on their common boundaries and define a partial measured foliation $\tilde{F}^*$ on $\tilde{S}$ which has the property that the group $\Gamma$ of covering translations of $\tilde{S} \to S$ acts affinely on this foliation, with holonomy
homomorphism defined by the class \( \varphi \) of the train track \( \sigma \) equipped with its tranverse orientation and its invariant measure. The partial foliation \( \tilde{F}^* \) (without its transverse measure) is a lift of the partial foliation \( F^* \) on \( S \). By collapsing each connected component of \( S - F^* \) onto a spine, we obtain a foliation \( F \) on \( S \), and performing the lift of these operations on the connected components of \( \tilde{S} - \tilde{F}^* \), we obtain a lift \( \tilde{F} \) of \( F \) which is equipped with a transverse measure coming from that of \( \tilde{F}^* \), on which the group \( \Gamma \) acts affinely with holonomy homomorphism \( \varphi \). Thus, we have an element of \( \mathcal{A}_F \varphi \). This element is well-defined from the broken measure on \((\tau, \sigma)\), up to the choice of the component \( S_0 \) that we started with. Making another choice has the effect of multiplying the resulting transverse measure of \( \tilde{F} \) by a constant. Summing up, we have a well-defined element of \( \mathcal{PA}_F \varphi \).

We denote by \( B(\tau, \sigma) \) the set of projective classes of broken measures on \((\tau, \sigma)\).

**Remarks:**

1. In the case where the train track \( \sigma \) is topologically a simple closed curve, the construction that we just described coincides with that of Oertel ([4] p. 308). In fact, Oertel describes a construction in a general setting where \( \sigma \) is a weighted transversely oriented condimension-1 submanifold representing a cohomology class in an \( n \)-dimensional manifold and \( \tau \) a branched codimension-1 manifold.

2. The map \( I_{\tau, \sigma} : B(\tau, \sigma) \to \mathcal{PA}_F \varphi \) which assigns to a broken measure on a fixed pair \((\tau, \sigma)\) the element of \( \mathcal{PA}_F \varphi \) defined by Construction 2.6 is not injective in general. However, in the case where \( \sigma \) is a simple closed curve and where no connected component of \( S - \tau \) is a disk with 0, 1 or 2 spikes on its boundary or an annulus with no spike on its boundary, then the map \( I_{\tau, \sigma} \) is injective, by Theorem 3.1 of [8]. In fact, following that proof, one can prove that the map \( I_{\tau, \sigma} \) is injective if no connected component of \( S - \tau \) is a disk with 0, 1 or 2 spikes on its boundary, without the assumption that \( \tau \) is topologically a simple closed curve. We shall not use this fact here.

**Proposition 2.7.** Let \( \sigma \) be a transversely oriented train track equipped with an invariant measure and let \( \varphi \) be the element it defines in \( H^1(S; \mathbb{R}) \). For each \( F \in \mathcal{AF} \varphi \), there exists a train track \( \tau \) which is transverse to \( \sigma \) such that the projective class \( F \) is obtained from a broken measure on \((\tau, \sigma)\) by Construction 2.6. This broken measure is well-defined up to a scalar multiple.

**Proof.** Let \( F \in \mathcal{AF} \varphi \). We represent \( F \) by a partial foliation \( F^* \) on \( S \), whose support is a fibered neighborhood \( V(\tau) \) of a train track \( \tau \) and which is transverse to the foliation of \( V(\tau) \) by the ties (Proposition 1.3). Let \( \tilde{F}^* \) and \( \tilde{\sigma} \) be respectively the lifts of \( F^* \) and \( \sigma \) to \( \tilde{S} \). By the definition of an affine foliation, the partial foliation \( \tilde{F}^* \) is equipped with an
invariant transverse measure. Choose a connected component $\tilde{F}_1$ of $\tilde{F}^* - \tilde{\sigma}$. Since the action of the group $\Gamma$ of covering translations on the transverse measure of $\tilde{F}^*$ is affine with holonomy homomorphism $\varphi$, the stabilizer of the component $\tilde{F}_1$ in $\Gamma$ preserves the transverse measure in this component, and $\tilde{F}_1$ projects in $S$ onto a component $F_1$ of $F^* - \sigma$ which is equipped with a well-defined transverse measure. From this measure, we can define in a natural way a set of weights on the edges of $(\tau, \sigma)$ which are the images of $F_1$ by the natural projection $V(\tau) \to \tau$. This set of weights can be completed in a unique manner into a broken measure on $(\tau, \sigma)$, which satisfies the required properties. This broken measure is well-defined up to the choice of the component $\tilde{F}_1$, and this makes the broken measure well-defined up to a scalar multiple.

**Definition 2.8 (Affine foliation carried by a pair).** We shall say that the element $F \in \overline{AF}_\varphi$ in Proposition 2.7 is carried by the pair $(\tau, \sigma)$. The equivalence class of $F$ is an element of the image set $\mathcal{I}_{\tau, \sigma}(B(\tau, \sigma)) \subset \overline{PAF}_\varphi$.

### 3. Topology

Broken measures can be used to define a topology on the space $\overline{PAF}$ of projective affine foliations. We recall first some set-theoretic terminology.

**Directed sets and inductive limits.** A relation $\prec$ on a set $E$ is said to be a preorder relation if it is reflexive and transitive. The set $E$, equipped with such a relation, is said to be a preordered set. A preordered set $(E, \prec)$ in which every finite subset has an upper bound (with respect to the relation $\prec$) is called a directed set.

Let $(I, \prec)$ be a directed preordered set and let $(E_\alpha)_{\alpha \in I}$ be a family of sets indexed by $I$ and satisfying the following properties:

(i) for each $(\alpha, \beta) \in I \times I$ with $\alpha \prec \beta$, there exists a map $f_{\beta\alpha}: E_\alpha \to E_\beta$;

(ii) for each $(\alpha, \beta, \gamma) \in I \times I \times I$ with $\alpha \prec \beta \prec \gamma$, we have $f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\alpha}$;

(iii) for each $\alpha \in I$, $f_{\alpha\alpha}: E_\alpha \to E_\alpha$ is the identity map.

Such a family $(E_\alpha)_{\alpha \in I}$ equipped with the set of maps $(f_{\beta\alpha}: E_\alpha \to E_\beta)$, is called an inductive system if sets (relatively to $I$).

Let $(E_\alpha)_{\alpha \in I}$ be an inductive system of sets and let $\overline{E} = \bigcup_{\alpha \in I} E_\alpha$ be the disjoint union of the family $(E_\alpha)_{\alpha \in I}$. We define an equivalence relation $\sim$ on $\overline{E}$. Given $x$ and $y \in \overline{E}$, let $\alpha$ and $\beta \in I$ be such that $x \in E_\alpha$ and $y \in E_\beta$. We say that $x \sim y$ if there exists $\gamma \in I$ such that $\alpha \prec \gamma \prec \beta \prec \gamma$ and $f_{\gamma\beta}(x) = f_{\gamma\alpha}(y)$. Then, $\sim$ is an equivalence relation on $\overline{E}$. The quotient set $\overline{E}/\sim$ is called the inductive limit of the family $(E_\alpha)_{\alpha \in I}$ with respect to the family of maps $(f_{\alpha\beta})_{\alpha \in I}$. 
Let \( f : E \to \overline{E} \) be the canonical map. For each \( \alpha \in I \), the restriction \( f_{\alpha} \) of \( f \) to \( E_{\alpha} \) is called the canonical map from \( E_{\alpha} \) to \( \overline{E} \). The various canonical maps satisfy the relations \( f_{\beta} \circ f_{\beta} = f_{\alpha} \) for all \( \alpha \) and \( \beta \) with \( \alpha \prec \beta \). The inductive limit of the family \( (E_{\alpha})_{\alpha \in I} \) is also denoted by \( \lim_{\rightarrow} (E_{\alpha}, f_{\alpha}) \), or simply \( \lim_{\rightarrow} E_{\alpha} \).

**Inductive limit topology.** Suppose now that for each \( \alpha \in I \), the space \( E_{\alpha} \) is a topological space, and that the maps \( f_{\beta} : E_{\alpha} \to E_{\beta} \) are continuous. Then, there is a natural topology on the set \( \lim_{\rightarrow} E_{\alpha} \), called the inductive limit topology. This is the quotient topology on \( \lim_{\rightarrow} E_{\alpha} \), seen as the quotient by the relation of the set \( \overline{E} = \bigcup_{\alpha \in I} E_{\alpha} \) equipped with the disjoint union topology. The inductive limit topology is also the weakest topology on the set \( \overline{E} \) which makes the canonical map \( f : \overline{E} \to \overline{E} \) continuous.

The canonical maps \( f_{\alpha} : E_{\alpha} \to \lim_{\rightarrow} E_{\alpha} \) are also continuous. The space \( \overline{E} = \lim_{\rightarrow} E_{\alpha} \) equipped with the family of continuous maps \( (f_{\alpha} : E_{\alpha} \to \lim_{\rightarrow} E_{\alpha})_{\alpha \in I} \) satisfies the following universal property: for every topological space \( X \) equipped with a family of continuous maps \( (g_{\alpha} : E_{\alpha} \to X)_{\alpha \in I} \), satisfying \( g_{\beta} \circ f_{\beta} = g_{\alpha} \) for all \( \alpha \) and \( \beta \) with \( \alpha \prec \beta \), there exists a unique continuous map \( g : \lim_{\rightarrow} E_{\alpha} \to X \) satisfying \( g_{\alpha} = g \circ f_{\alpha} \) for all \( \alpha \in I \).

In what follows, we take \( I \) to be the set of isotopy classes of pairs \((\tau, \sigma)\), where
(i) \( \tau \) is a train track on \( S \);
(ii) \( \sigma \) is an oriented train track on \( S \);
(iii) \( \tau \) and \( \sigma \) intersect transversely;
(iv) no intersection point of \( \tau \) and \( \sigma \) is a switch of \( \tau \) or of \( \sigma \).

A pair of train tracks \((\tau, \sigma)\) satisfying conditions (iii) and (iv) above is said to be in general position.

**Definition 3.1 (Fibered neighborhood of a pair).** Let \((\tau, \sigma)\) be a pair of train tracks in general position. A fibered neighborhood \( N(\tau, \sigma) \) of \((\tau, \sigma)\) is a union \( N(\tau) \cup N(\sigma) \) of fibered neighborhood of \( \tau \) and \( \sigma \), such that each connected component of \( N(\tau) \cap N(\sigma) \) is a rectangle on which the foliations of \( N(\tau) \) and \( N(\sigma) \) by the ties induce a product foliation, as indicated in Figure 11. There is an inclusion \( \tau \cap \sigma \subset N(\tau, \sigma) \) which is well-defined up to isotopy, and the connected components of \( N(\tau) \cap N(\sigma) \) are in natural one-to-one correspondence with the points of \( \tau \cap \sigma \) (each such component being a neighborhood of the corresponding point of \( \tau \cap \sigma \)). There is a natural map \( N(\tau, \sigma) \to \tau \cup \sigma \), which is obtained by collapsing \( N(\tau) \) and \( N(\sigma) \) on \( \tau \) and \( \sigma \) respectively, along the fibers of the neighborhood \( N(\tau) \) and \( N(\sigma) \) respectively. Clearly, the collapses can be done in any order.

We define now a relation \( \prec \) on \( I \). We first say that \((\tau_1, \sigma_1) \prec (\tau_2, \sigma_2)\) if there exists a homeomorphism \( f : S \to S \) which is isotopic to the identity such that \( f(\tau_1 \cap \sigma_1) \) is contained in a fibered neighborhood \( N(\tau_2, \sigma_2) \) of \((\tau_2, \sigma_2)\), with \( f(\tau_1) \) transverse to the ties of \( N(\tau_2) \) and
Proposition 3.2. The relation $(\tau_1, \sigma_1) \prec (\tau_2, \sigma_2)$ induces a natural map $B(\tau_1, \sigma_1) \to B(\tau_2, \sigma_2)$ which, at the level of projective affine foliations, is the inclusion map.

Proof. Let $f : S \to S$ be a homeomorphism isotopic to the identity such that $f(\tau_1 \cup \sigma_1)$ is contained in a fibered neighborhood $N(\tau_2, \sigma_2)$ of $(\tau_2, \sigma_2)$, with $f(\tau_1)$ transverse to the ties of $N(\tau_2)$ and $f(\sigma_1)$ transverse to the ties of $N(\sigma_2)$.

This homeomorphism $f$ establishes a natural one-to-one correspondence between the broken measures on $(\tau_1, \sigma_1)$ and the broken measures on $(f(\tau_1), f(\sigma_1))$. Since $f$ is isotopic to the identity, two such corresponding broken measures define the same element of $\overline{PAF}$. Thus, we can assume without loss of generality that $(\tau_1, \sigma_1)$ is contained in $N(\tau_2, \sigma_2)$, with $\tau_1$ transverse to the ties of $N(\tau_2)$, and $\sigma_1$ transverse to the ties of $N(\sigma_2)$.

Let $\mu$ be now a broken measure on such a pair $(\tau_1, \sigma_1)$ and let us see how $\mu$ induces a system of weights on the edges of the pair $(\tau_2, \sigma_2)$. This is done in three steps:

Step 1. By performing an isotopy of $\tau_1$ supported in a neighborhood of $N(\tau_2) \cap N(\sigma_2)$, we replace $\tau_1$ by a train track $\tau'_1$ such that the new pair $(\tau'_1, \sigma_1)$ satisfies all the properties of the pair $(\tau_1, \sigma_1)$, with the further property that no switch of $\tau'_1$ is contained in $N(\sigma_2)$. This isotopy is described in Figure 12. We note that the isotopy class of the pair $(\tau'_1, \sigma_1)$ may be different from that of the pair $(\tau_1, \sigma_1)$; this modification of the isotopy class of pairs is done by a finite number of moves which are of the type described in Figure 13, or the inverse move. It is easy to see that each of these moves induces a natural map between the broken measures on the pairs $(\tau_1, \sigma_1)$ and $(\tau'_1, \sigma'_1)$, which is the identity map at the level of the space $\overline{PAF}$. Thus, we can suppose without loss of generality that there are no switches of $\tau'_1$ in $N(\sigma_2)$.
Step 2. The inclusion of $\sigma_1$ in $N(\sigma_2)$, transversely to the ties, induces a linear map from the space of invariant measures on $\sigma_1$ to the space of invariant measures on $\sigma_2$, which is defined as follows: an invariant measure on $\sigma_1$ induces a real number on each tie of $N(\sigma_2)$ (by counting the intersections with multiplicity), and this collection defines a weight on each edge $e$ of $\sigma_2$ by associating to $e$ the real number associated to the tie which lies above it. It is easy to see that this weight on $e$ does not depend on the choice of the tie, and that the resulting system of weights on the edges of $\sigma_2$ constitutes an invariant measure. It is also clear that since the orientations of $\sigma_1$ and $\sigma_2$ induce the same transverse orientation on each tie of $N(\sigma_2)$, the cohomology class of the oriented train track $\sigma_1$ equipped with its invariant measure is the same than that of the oriented train track $\sigma_2$ equipped with the resulting invariant measure.

Collapsing then the neighborhood $N(\sigma_2)$ of $\sigma_2$ along the ties of $N(\sigma_2)$ transforms the broken measure $\mu$ on $(\tau_1, \sigma_1)$ into a broken measure on the pair $(\tau_1, \sigma_2)$, and the induced map on the spaces of broken measures is the identity map at the level of the space $\overline{\mathcal{PAF}}$.

Step 3. The broken measure on $(\tau_1, \sigma_1)$ induces now in a natural manner a broken measure on $(\tau_2, \sigma_2)$, by assigning to each edge of $(\tau_2, \sigma_2)$ the total weight of a tie which lies above it, as in the construction we described in Step 2, applied now to edges of pairs.
It is clear here also that the induced map on $\mathcal{PAF}$ is the identity map. This proves Proposition 3.2.

**Proposition 3.3.** The relation $\prec$ is a preorder relation on $I$, and the preordered set $(I, \prec)$ is directed.

**Proof.** The fact that the relation is a preorder relation is clear. Let us show that the preordered set $(I, \prec)$ is directed. Let $(\tau_1, \sigma_1), \ldots, (\tau_n, \sigma_n)$ be pairs of train tracks representing elements of $I$. By taking, for each $i = 1, \ldots, n$, an appropriate homeomorphism $f_i: S \to S$ which is isotopic to the identity, and by replacing each $(\tau_i, \sigma_i)$ by its image $(f_i(\tau_i), f_i(\sigma_i))$, we can assume without loss of generality that for each $i$ and $j \in \{1, \ldots, n\}$ with $i \neq j$, the pair $(\tau_i, \sigma_i)$ is transverse to the pair $(\tau_j, \sigma_j)$. We can smooth then each intersection point of $\tau_i$ with $\tau_j$, as indicated in Figure 14, in such a way that the union $\tau_1 \cup \ldots \cup \tau_n$ is naturally a train track, which we call $\tau$, in such a way that $\tau_i \prec \tau$ for each $i = 1, \ldots, n$. Each such intersection point gives rise to two switches of $\tau$. We note that there are two locally distinct ways of smoothing each intersection point in such a way that the train tracks before the operation are carried by the resulting train track, and we choose at each point one of these ways, in an arbitrary manner. We then perform operations of the same kind on the train tracks $\sigma_1, \ldots, \sigma_n$, but here we smooth the intersection points in a way compatible with the orientations (Figure 15). We let $\sigma$ be the train track obtained as the union of the modified $\sigma_i$’s. It is clear that for every $i = 1, \ldots, n$, we have $(\tau_i, \sigma_i) \prec (\tau, \sigma)$. This proves Proposition 3.3.

![Fig. 14. Smoothing a point of $\tau_i \cap \tau_j$.](image)

![Fig. 15. Smoothing a point of $\sigma_i \cap \sigma_j$.](image)
We now consider the preordered set \( I \) defined by Proposition 3.3. For each \( \alpha \in I \), we take a pair \((\tau, \sigma)\) representing \( \alpha \), and we let \( e_1, \ldots, e_N \) be the edges of this pair. Let \( \mathbb{R}^N \) be the real vector space with basis \((e_1, \ldots, e_N)\). The set of broken measures on \((\tau, \sigma)\) is naturally a convex cone in \( \mathbb{R}^N \). The set \( B(\tau, \sigma) \) of projective classes of such broken measures is the projectivization of this convex cone, and as such it is equipped with a natural topology. This projectivized convex cone, as a topological space, is well-defined out of the element \( \alpha \in I \) (that is, independently of the choice of the pair \((\tau, \sigma)\) representing it). We also recall that for \((\tau, \sigma)\) representing \( \alpha \in I \), there is a map \( I_{\tau, \sigma} : B(\tau, \sigma) \to \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_\alpha \), and we denote by \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\tau, \sigma} \subset \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_\alpha \) the image set \( I_{\tau, \sigma}(B(\tau, \sigma)) \). We note here also that the definition of the set \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\tau, \sigma} \) depends only on the element \( \alpha \in I \), and not on the element \((\tau, \sigma)\) representing it. Thus, we can denote the space \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\tau, \sigma} \) as \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_\alpha \). We note by the way that the map \( I_{\tau, \sigma} \) is not always injective. We equip the space \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_\alpha \) with the image topology (that is, the quotient topology with respect to the map \( I_{\tau, \sigma} \) and the topology of \( B(\tau, \sigma) \)). By Proposition 3.2, if \((\tau_1, \sigma_1) \prec (\tau_2, \sigma_2)\) represent two elements \( \alpha_1 \) and \( \alpha_2 \) of \( I \) satisfying \((\tau_1, \sigma_1) \prec (\tau_2, \sigma_2)\), then there is an induced map \( B(\tau_1, \sigma_1) \to B(\tau_2, \sigma_2) \). It is easy to see that this map is continuous, and that it induces a well-defined map \( f_{\alpha_2 \alpha_1} : \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha_1} \to \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha_2} \) which, by Proposition 3.2, is the inclusion map. This gives the following

**Proposition 3.4.** The family \((\overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha})_{\alpha \in I}\) equipped with the set of maps \((f_{\alpha_2 \alpha_1})_{\alpha \in I}\), is a directed family of topological spaces.

**Proposition 3.5.** There is a natural identification between the inductive limit of the family \((\overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha})_{\alpha \in I}\) and the set \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}} \) of projective classes of affine foliations on \( S \).

**Proof.** For each \( \alpha \in I \), there is a natural map \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha} \to \overline{\mathcal{P}_\mathcal{A}\mathcal{F}} \), which in fact is the inclusion map, and the union of all these maps, for the various \( \alpha \)'s, gives a map from the disjoint union \( \bigcup_{\alpha \in I} (\overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha})_{\alpha \in I} \) to \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}} \). From the definition of the equivalence relation \( \sim \) defining the inductive limit, \( \bigcup_{\alpha} (\overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha})_{\alpha \in I} \sim \lim_\to (\overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha})_{\alpha \in I} \) there is an induced map \( \lim_\to (\overline{\mathcal{P}_\mathcal{A}\mathcal{F}}_{\alpha})_{\alpha \in I} \to \overline{\mathcal{P}_\mathcal{A}\mathcal{F}} \), and that this map is injective. By proposition 1.3, this maps is also onto.

**The topology of \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}} \).** We equip the set \( \overline{\mathcal{P}_\mathcal{A}\mathcal{F}} \) with the inductive limit topology, using the natural map provided by Proposition 3.5.

We consider now the subspace \( \mathcal{P}_\mathcal{A}\mathcal{F} \subset \overline{\mathcal{P}_\mathcal{A}\mathcal{F}} \) consisting of the projective equivalence classes of affine foliations whose singular points are \( n \)-prongs with \( n \geq 3 \). This is the set of equivalence classes considered by Hatcher and Oertel in [2]. In that paper, the authors
equip the space $\mathcal{PAF}$ with a natural topology which is defined by embedding this space in a space of functions (defined by integrating transverse measures on curves) on the set of equivalence classes of closed curves in the universal abelian cover of $S$. We shall call this topology the Hatcher-Oertel topology. It is natural to compare the two topologies on $\mathcal{PAF}$. We have the following:

**Theorem 3.6.** The topology induced on the set $\mathcal{PAF}$ by the inclusion $\mathcal{PAF} \subset \overline{\mathcal{PAF}}$ coincides with the Hatcher-Oertel topology.

**Proof.** This is a consequence of Theorem 3.2 of [8] which says that broken measures on train tracks provide local coordinates for the topology of $\mathcal{PAF}$. More precisely, if $\tau$ is a train track with the property that no connected component of $S - \tau$ is a disk with 0 or 1 or 2 spikes on its boundary or an annulus with no spike on its boundary, and if $C$ is an oriented train track which is topologically a simple closed curve, equipped with a nonnegative weight, then the associated map $\mathcal{J}_{\tau,C} : B(\tau, C) \to \mathcal{PAF}$ is a homeomorphism onto its image, with $\mathcal{PAF}$ being equipped with the Hatcher-Oertel topology. Theorem 3.3 of [8] says furthermore that if $\tau$ is a recurrent train track (i.e. if there exists an invariant measure on $\tau$ with all weights $> 0$) with the property that each component of $S - \tau$ is a disk with exactly 3 spikes on its boundary, then the image of the map $\mathcal{J}_{\tau,C}$ is a top-dimensional subspace of $\mathcal{PAF}$. □

It is an interesting question to study the topological properties of the various spaces introduced. We recall that by the result of [2], for each holonomy homomorphism $\varphi$, the space $\overline{\mathcal{PAF}}_{\varphi}$ is homeomorphic to the sphere $S^{6g-7}$, and the space $\mathcal{PAF}$ is homeomorphic to $S^{6g-7} \times H^1(S; \mathbb{R})$. The spaces $\overline{\mathcal{PAF}}_\varphi$ and $\overline{\mathcal{PAF}}$ are of course infinite dimensional. We have the following:

**Proposition 3.7.** The space $\overline{\mathcal{PAF}}$ is arcwise connected, and for every holonomy homomorphism $\varphi$, the space $\overline{\mathcal{PAF}}_\varphi$ is arcwise connected.

**Proof.** It suffices to take sums of pairs of train tracks, as in the proof of Proposition 3.3 (the operations described in Figures 14 and 15), and to take convex combinations of broken measures on the resulting sums. □

In [7], we considered the subset of measured foliations space consisting of equivalence classes of measured foliations which can be represented by orientable partial measured foliations (or equivalently, by orientable measured laminations), and we proved that this space is arcwise connected. Likewise, we can consider the subset $\overline{\mathcal{OPAF}}$ (respectively $\overline{\mathcal{OPAF}}_\varphi$) of $\overline{\mathcal{PAF}}$ (respectively $\overline{\mathcal{PAF}}_\varphi$) whose elements are projective equivalence classes of affine foliations (respectively, affine foliations with fixed holonomy...
homomorphism $\varphi$) which can be represented by orientable partial affine foliations. Equivalently, an element of $\overline{\text{OPAF}}$ (respectively $\overline{\text{OPAF}_\varphi}$) is the equivalence class of an affine foliation which is carried by a pair $(\tau, \sigma)$ (using the notations of Definition 2.8) with $\tau$ orientable.

By the same proof as for Proposition 3.7, applied now to oriented pairs of train tracks, we obtain the following

**Proposition 3.8.** The space $\overline{\text{OPAF}}$ is arcwise connected, and for every holonomy homomorphism $\varphi$, the space $\overline{\text{OPAF}_\varphi}$ is arcwise connected. ■

Note that the proof of Proposition 3.8 is simpler than that of the analogous result of [7], which involved considerations on the complementary components of the train tracks.

**References**